And... action! - Monoid Acts and (Pre)orders

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Beginner's Course in Computer Science: Orders

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Week 42 All are

$$x \sqsubseteq y \quad :\Longleftrightarrow \quad \exists m \in M : m \otimes x = y,$$

where M is a set that "acts" on a set A via $\otimes$.

Monoid Acts

Definition

Let $(M, *, e)$ be a monoid ($*$ is associative with neutral element e).
$\otimes : M \to A \to A$ is called *monoid act* if and only if

- $e \otimes - = id$
- $\forall x, y \in M : \forall a \in A : \quad x \otimes (y \otimes a) = (x * y) \otimes a$

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Examples

set A	monoid	act $\otimes$
carrier of a monoid M	$(M, *, e)$	$*$
$\mathbb{Z}$	$(\mathbb{N}, +, 0)$	$+ : \mathbb{N} \to \mathbb{Z} \to \mathbb{Z}$
Q where (Q, Σ, δ) is a transition system	$(\Sigma^\star, +\!\!+, \varepsilon)$	*flip* $\delta^\star$

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For all $x, y \in A$:

$$x \sqsubseteq_{A,\, M,\, \otimes} y :\Longleftrightarrow \exists m \in M : m \otimes x = y\,.$$

Orders Revisited

order	$\sqsubseteq_{A,\,M,\,\otimes}$
$\leq$	$\sqsubseteq_{\mathbb{N},\,\mathbb{N},\,+}$
$\trianglelefteq$	$\sqsubseteq_{A^\star,\,A^\star,\,\mathit{flip}\,(+\!\!+)}$
$\mid$	$\sqsubseteq_{\mathbb{Z},\,\mathbb{N},\,\cdot\mid_{\mathbb{N}}}$

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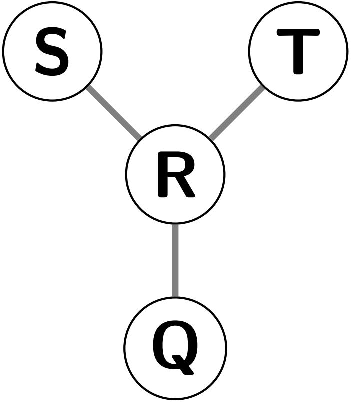
Question 1. How expressive is this abstraction?

Question 2. What are the properties of $\sqsubseteq_{A,\,M,\,\otimes}$?

Answer 1: Expressiveness is universal

Idea: Build a transition system from an order



Hasse diagram

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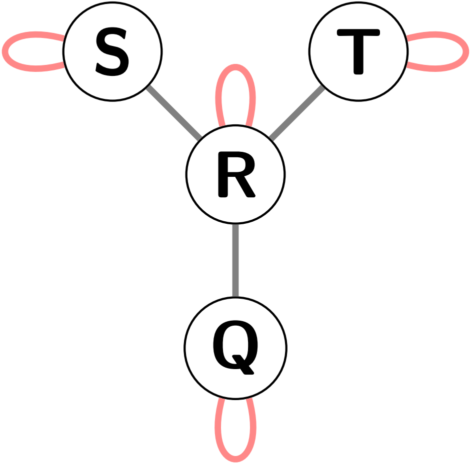

Hasse diagram + reflexivity

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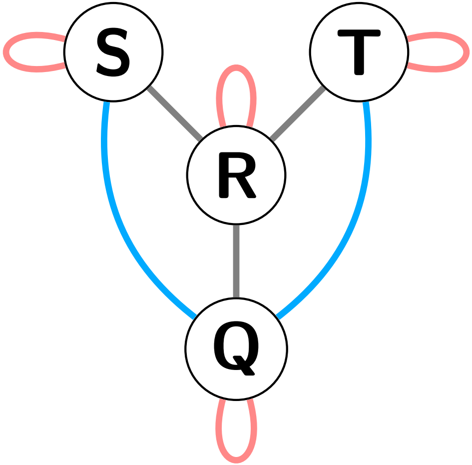

Hasse diagram + reflexivity + transitivity

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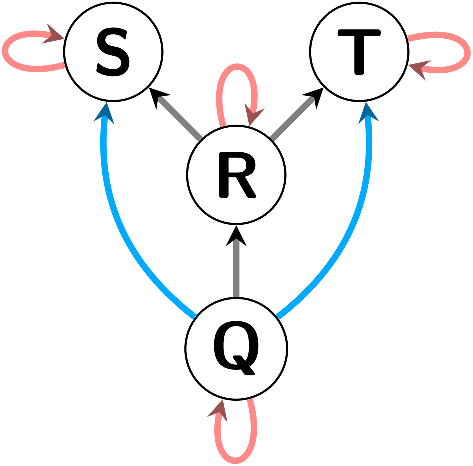

Hasse diagram + reflexivity + transitivity + directions

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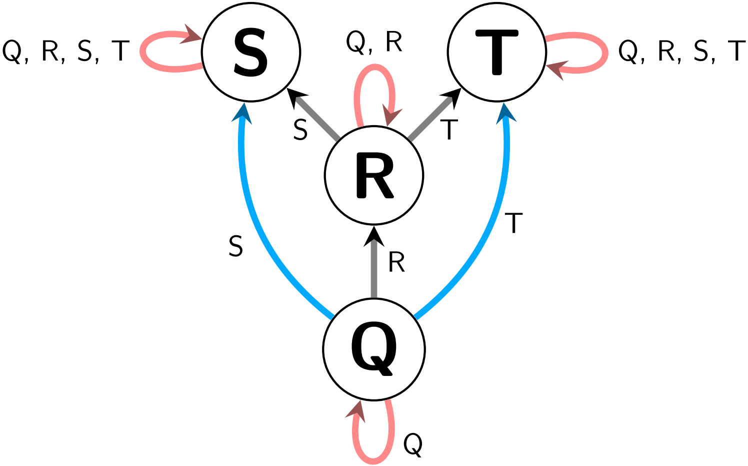


Transition system

Answer 1: Formalism

Preconditions

Given: (pre)order $(A, \preccurlyeq)$
Define:

$$\begin{aligned} Q &:= A \\ \Sigma &:= A \\ \delta &: Q \to \Sigma \to Q, \\ & q\ \varsigma \mapsto \begin{cases} \varsigma & : q \preccurlyeq \varsigma \\ q & : \text{otherwise} \end{cases} \end{aligned}$$

Property I

For all $x, y \in A$:

$$x \preccurlyeq y \iff \delta\ x\ y = y$$

$$(\text{for sets: } X \subseteq Y \iff X \cup Y = Y)$$

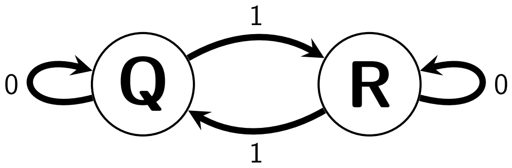
Property II

(Q, Σ, δ) is a transition system and

$$\sqsubseteq_{Q,\, \Sigma^\star,\, \mathit{flip}\ \delta^\star} \quad = \quad \preccurlyeq$$

Answer 2: Properties of $\sqsubseteq_{A,M,\otimes}$

Always a preorder (reflexive and transitive)
Generally *not* antisymmetric:



Cycles are a problem – even *the* problem

Answer 2: Characterisation of antisymmetry

Theorem

The following statements are equivalent:

1. $\sqsubseteq_{A,M,\otimes}$ is antisymmetric.
2. The corresponding transition system has no non-trivial cycles.
3. $Fix \circ \otimes : (M, *, e) \to (2^A, \cap, A)$ is monoid homomorphism.[1]

[1] For $f : A \to A$ we have $Fix(f) = \{ a \in A \mid f\ a = a \}$

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- Surprisingly simple in applications
- Sufficient conditions even more simple
- Proof $\widehat{=}$ every antisymmetry proof

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Application: Implementation

$\sqsubseteq_{A,M,\otimes}$	parametric	$\rightsquigarrow$	functional programming
$\exists m \in M : m \otimes a = b$	logic	$\rightsquigarrow$	logic programming

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WFLP	in Kiel	$\rightsquigarrow$	Curry

Application: Suffix list

Consider the "has-suffix"-relation $\trianglerighteq$:

$$\begin{aligned} xs \trianglerighteq ys &\iff \exists ms : xs = ms +\!\!+\, ys && \text{(not in prev. form}^2\text{)} \\ &\iff \exists m \in \mathbb{N} : drop\; m\; xs = ys, && \text{(in prev. form)} \end{aligned}$$

where $drop : \mathbb{N} \to A^\star \to A^\star$.

drop is an act

- $drop\; 0\; xs = xs$
- $drop\; m\; (drop\; n\; xs) = drop\; (m+n)\; xs$.

drop has the "no-cycles-property" (on *finite, deterministic* arguments)

[2] Require $x \preccurlyeq y \iff \exists m : m \otimes x = y$.

Application: Suffix list in Curry

$$
\begin{array}{l}
(\trianglerighteq) :: [\alpha] \to [\alpha] \to \mathit{Success} \\
xs \trianglerighteq ys = \mathit{drop}\ m\ xs =:= ys \\
\quad \textbf{where}\ m\ \textbf{free}
\end{array}
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General version:

$$x \sqsubseteq_{A,\, M,\, \otimes} y \iff \exists m \in M : m \otimes x = y$$

$$
\begin{array}{l}
\mathit{relatedBy} :: (\mu \to \alpha \to \alpha) \to \alpha \to \alpha \to \mathit{Success} \\
\mathit{relatedBy}\ (\otimes)\ x\ y = m \otimes x =:= y \\
\quad \textbf{where}\ m\ \textbf{free} \\
(\trianglerighteq) :: [\alpha] \to [\alpha] \to \mathit{Success} \\
(\trianglerighteq) = \mathit{relatedBy}\ \mathit{drop}
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So...

theoretically

- algebraic structure $\leftrightarrow$ relational concept
- allows application of additional tools
- reveals inner structure

practically

- modularity, declarative style
- fits into functional logical paradigm
- simple implementation

Thank you for listening!

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Enjoy your lunch.